

LETTER TO THE EDITOR

Point perturbations of circle billiards

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Abstract. The spectral statistics of the circular billiard with a point-scatterer is investigated. In the semiclassical limit, the spectrum is demonstrated to be composed of two uncorrelated level sequences. The first corresponds to states for which the scatterer is located in the classically forbidden region and its energy levels are not affected by the scatterer in the semiclassical limit while the second sequence contains the levels which are affected by the point-scatterer. The nearest neighbor spacing distribution which results from the superposition of these sequences is calculated analytically within some approximation and good agreement with the distribution that was computed numerically is found.

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Classical dynamics may be illuminating for the understanding of the corresponding quantum mechanical systems. One of the most studied aspects of the relation between classical and quantum mechanics is the connection between the spectral statistics of the quantum system and the dynamical properties of its classical counterpart. Classically integrable systems typically exhibit Poisson-like spectral statistics [1] while classically chaotic systems exhibit spectral statistics of random matrix ensembles [2, 3, 4, 5]. The spectral statistics of integrable and chaotic systems are universal, that is, they do not depend on specific details of the system but rather on the type of motion and its symmetries. There are systems which are intermediate between integrable and chaotic ones and their spectral properties are not known to be universal. Such systems are of experimental relevance. The spectral statistics of mixed systems, for which the phase space is composed of both integrable and chaotic regions, were studied by Berry and Robnik [6]. The spectrum can be viewed as a superposition of uncorrelated level sequences, corresponding to the various regions, which are either chaotic or integrable. The nearest neighbor spacing distribution (NNSD) of such a superposition of sequences was calculated in [6]. The resulting statistics are, in some sense, intermediate between those of integrable and chaotic systems. Other types of systems with intermediate statistics include pseudointegrable systems and integrable systems with flux lines or point-scatterers [7, 8, 9, 10, 11, 12, 13].

The spectral statistics of billiards with flux lines [7,8] and of some pseudointegrable billiards [8] were recently studied. A possible route towards an understanding of the spectral statistics of these systems is based on their classical periodic orbits. It is possible to compute the correlation function of the energy levels from these orbits by using trace formulae [14,15]. For the billiards with flux lines or for pseudointegrable billiards, the orbits include not only the periodic orbits but also diffracting orbits which are built from segments that start and end at some singularity of the system. While the contributions from periodic orbits were easily calculated, those from diffracting orbits turn out to be much more involved. The spectral statistics of these systems, that can be obtained numerically, appear to be intermediate between those of integrable and chaotic systems. In particular, the NNSD show level repulsion at small spacings and an exponential falloff at large spacings.

Since the contributions of diffracting orbits to the spectral statistics of pseudointegrable systems and of billiards with flux lines is far from being understood, it is of interest to study simpler systems which exhibit intermediate statistics. A class of such systems is given by integrable systems with a point-scatterers. A point-scatterer is the self adjoint extension of a “ δ -function potential” in two or three dimensions [16]. The spectral statistics of an integrable system with such a point-scatterer was first studied by Šeba [10]. It is a rectangular billiard with the perturbation at its center. This “Šeba billiard” also exhibits intermediate statistics which differs from that of psedointegrable systems. Integrable systems with point-scatterers are much easier to study analytically compared to integrable systems with flux lines or to pseudointegrable systems. The contributions of diffracting orbits to the correlation function of the energy levels were recently calculated for the rectangular billiard with a point-scatterer [11,12]. Exact results for the NNSD were also obtained [13]. One of the intriguing features of the spectral statistics of the rectangular billiard with a point-scatterer (and Dirichlet boundary conditions) is its dependence on the location of the scatterer. If the coordinates of the scatterer (divided by the sides of the rectangle) are rational numbers p_i/q_i (where $i = x, y$) then the spectral statistics depend in a non trivial way on p_i, q_i [13]. In contrast, for typical locations, the spectral statistics seem to be location independent. The cause for this dependence on location is that many wavefunctions vanish at rational values of the coordinates. At these locations there are many degeneracies in the lengths of the diffracting orbits (including repetitions). Since such dependence on location is atypical, it is of interest to study the dependence of the spectral statistics on the location of the perturbation in other systems. For instance, it may be possible that for typical systems the location of the point scatterer affects the spectral statistics only smoothly (and does not depend on the rationality of the coordinates of the scatterer).

The system that is studied in this work is the circle billiard perturbed by a point-scatterer and the dependence of the spectral statistics on its location is studied. The (two dimensional) circle billiard with radius R is described by the Schrödinger equation

$$-\Delta\psi = E\psi \quad (1)$$

with Dirichlet boundary conditions $\psi(|\mathbf{r}| = R) = 0$. (The units where $\hbar = 2m = 1$ are used in most of this work.) The Hamiltonian with the point-scatterer is the self-adjoint extension of a Hamiltonian where one point, say \mathbf{x}_0 , is removed from its domain. It can be considered as the self-adjoint extension of a Hamiltonian with a δ -function potential at \mathbf{x}_0 . Given the eigenvalues (and eigenfunctions) of the unperturbed system E_n (and ψ_n), namely the system in absence of the δ -scatterer, the eigenvalues of the system with the point-scatterer are given by the roots of [17, 10]

$$\left(\frac{\sin \xi}{1 - \cos \xi} \right) \sum_n |\psi_n(\mathbf{x}_0)|^2 \frac{\Lambda}{\Lambda^2 + E_n^2} - \sum_n |\psi_n(\mathbf{x}_0)|^2 \left(\frac{1}{z - E_n} + \frac{E_n}{E_n^2 + \Lambda^2} \right) = 0, \quad (2)$$

where Λ and ξ are two parameters. For a more complete discussion regarding this equation, the roles of the parameters as well as the method of its numerical solution see, for example, [11]. This equation turns out to be very convenient for numerical solution since every root is located between two eigenvalues of the unperturbed system.

The (unnormalized) eigenfunctions of the circle billiard are

$$\tilde{\psi}_{n,m} = e^{\pm im\phi} \mathcal{J}_m(k_{n,m}r) \quad (3)$$

where \mathcal{J}_m are Bessel functions of the first kind and the angular momentum m is any nonnegative integer. The energy levels $E_{n,m} = k_{n,m}^2$ are determined by the boundary condition $\mathcal{J}_m(k_{n,m}R) = 0$. It is obvious that all the energy levels with $m \neq 0$ are doubly degenerate. As a result there is a linear combination of the two degenerate wavefunctions which vanishes at \mathbf{x}_0 and an orthogonal linear combination which does not vanish at \mathbf{x}_0 . The perturbation breaks this degeneracy. The linear combination which vanishes is also an eigenfunction of the Hamiltonian *with* the point-scatterer and thus $E_{n,m}$ is an eigenvalue of the perturbed problem. Therefore, half of the spectrum is unchanged by the perturbation. To avoid this trivial part of the spectrum we choose to work with the non vanishing linear combinations (which are eigenfunctions of the unperturbed Hamiltonian) and only the half of the spectrum which is affected by the perturbation will be considered in this work. For convenience, the location of the perturbation is chosen at $\mathbf{x}_0 = (r_0, 0)$ and therefore the eigenfunctions of the unperturbed Hamiltonian which do not vanish there are

$$\psi_{n,m}(\phi, r) = \sqrt{2/\pi(1 + \delta_{m,0})} (R\mathcal{J}_{m+1}(k_{n,m}R))^{-1} \cos m\phi \mathcal{J}_m(k_{n,m}r). \quad (4)$$

The spectrum is determined by substituting these eigenfunctions and the corresponding energies $E_{n,m} = k_{n,m}^2$ in equation (2).

We are interested in the dependence of the spectral statistics on the location of the scatterer. This dependence can be easily understood in terms of the properties of the wavefunctions of the unperturbed system. The quantum numbers n, m correspond to a state with an angular momentum $L = \hbar m$ and an energy $E_{n,m} = \hbar^2 k_{n,m}^2 / 2m$. In the semiclassical limit where $\hbar \rightarrow 0$, but E and L are kept fixed, the values of the wavefunctions are small in the classically forbidden region $r < r_{\min} = L/\sqrt{2mE}$. Therefore, one can divide the states into two groups. The first consists of states for which the point-scatterer is located in the classically forbidden region. As will

be demonstrated, the eigenvalues of these states change only slightly due to the perturbation (and do not change at all in the semiclassical limit). The second group includes the states for which the perturbation is located in the classically allowed region. These states will be (strongly) affected by the perturbation. This separation of the spectrum into a superposition of two sequences is the cause for the dependence of the spectral statistics on the location of the scatterer. This separation into (semiclassically) affected and unaffected states is justified in the following.

To demonstrate that some of the eigenvalues are almost unchanged by the perturbation one should solve equation (2) and show that for (exponentially) small eigenfunctions the corresponding eigenvalues are almost unaffected. For simplicity, instead of equation (2) it is sufficient to consider the finite sum

$$\sum_{i=1}^N \frac{|\psi_i|^2}{z - E_i} = a(z) \quad (5)$$

where $a(z)$ is assumed to be slowly varying function of z . We denote $\psi_i(\mathbf{x}_0) \equiv \psi_i$. To further simplify the argument let us assume that $|\psi_l|^2 = A$ and $|\psi_m|^2 = B$ are of order unity while for $i \neq l, m$ the wavefunctions on the scatterer $|\psi_i|^2 \equiv \epsilon_i$ are all small. The solutions of (5) are to a good approximation given by the solutions of

$$\frac{A}{z_0 - E_l} + \frac{B}{z_0 - E_m} = a(z_0) \quad (6)$$

for z_0 and by

$$z_i = E_i \quad (7)$$

for $i \neq l, m$. This is true since substituting a solution of the form $\tilde{z}_0 = z_0 + \delta z_0$ in (5) leads to

$$\frac{A}{z_0 + \delta z_0 - E_l} + \sum_{\substack{i=1 \\ i \neq l, m}}^N \frac{\epsilon_i}{z_0 + \delta z_0 - E_i} + \frac{B}{z_0 + \delta z_0 - E_m} = a(z_0 + \delta z_0). \quad (8)$$

Expanding with respect to δz_0 and then solving to the leading order in ϵ_i results in

$$\delta z_0 \simeq \left(\frac{A}{(z_0 - E_l)^2} + \frac{B}{(z_0 - E_m)^2} + a'(z_0) \right)^{-1} \sum_{\substack{i=1 \\ i \neq l, m}}^N \frac{\epsilon_i}{z_0 - E_i}. \quad (9)$$

When $\epsilon_i \rightarrow 0$, as is the case in the semiclassical limit, δz_0 also approaches 0. For the other eigenenergies one can substitute $z_i = E_i + \delta z_i$ (with $i \neq l, m$) and find that in the leading order

$$\delta z_i = \epsilon_i \left(a(E_i) - \frac{A}{E_i - E_l} - \frac{B}{E_i - E_m} \right)^{-1} \quad (10)$$

which also vanish when $\epsilon_i \rightarrow 0$. Note that we have just found $N - 1$ (approximate) solutions which are all the solutions between E_1 to E_N . It is not hard to generalize this calculation for more wavefunctions which are of order unity. In the semiclassical limit, the resulting spectrum always consist of such affected and unaffected components. In this limit the values of the wavefunctions at the scatterer are exponentially small if it

is located in the classically forbidden region and the corresponding eigenvalues can be treated as unchanged by the perturbation for any semiclassical consideration.

Consider a circle billiard with the point-scatterer at $r = r_0$. The spectral statistics of its energy levels in an energy window of width ΔE around E_0 are studied in what follows. Assume that E_0 is very large compared to ΔE , i.e. that all the levels in the window have similar energies which are high enough to be considered semiclassical. A natural question to ask is how many of these levels are affected by the point-scatterer and how many are unaffected by it. As was argued, the semiclassically unaffected levels are those for which the classical turning point, r_{min} , satisfies $r_{min} > r_0$. Equivalently, for a given energy E_0 , for a state to be unaffected, the angular momentum L should satisfy $L > L_0 = r_0\sqrt{2mE_0}$ (for the estimate of L_0 , the values of the energies are approximated by E_0). The fraction of such states was calculated in [18] where it was used to determine how many levels are poorly approximated in the WKB method. This fraction is

$$X_{un} = \frac{2}{\pi} \left[\cos^{-1} \left(\frac{r_0}{R} \right) - \frac{r_0}{R} \sqrt{1 - \frac{r_0}{R}} \right]. \quad (11)$$

It is clear that when $r_0 \rightarrow 0$ then $X_{un} \rightarrow 1$, as expected, while for $r_0 \rightarrow R$, $X_{un} \rightarrow 0$. As one approaches the semiclassical limit these states are less and less affected. The predictions of equation (11) can be checked numerically. The number of unaffected levels was calculated for two energy windows as a function of the location of the perturbation. The results are presented in figure 1. The radius of the billiard was chosen so that the mean level spacing is 1. Therefore both energy windows contain 20000 levels. A level was counted as unaffected if its difference from an energy level of the unperturbed system was less than 10^{-4} of the mean level spacing. This criterion is somewhat arbitrary, since in the semiclassical limit the difference can be taken to be arbitrarily small. Figure 1 indicates that equation (11) correctly describes the number of unaffected levels. There is a slight deviation which is smaller for the levels from the higher energy window. This deviation is caused by the fact that for any finite energy the wavefunctions do not vanish at the turning point r_{min} but rather exhibit an Airy-like structure (in the radial direction) near the turning point. This means that for states with $r_{min} > r_0$, for which r_{min} is close to r_0 , $|\psi(\mathbf{x}_0)|^2$ might not be small at a finite (but large) energy. The deviations are expected to vanish in the semiclassical limit as indicated by figure 1.

The spectrum of the circle billiard with a point-scatterer can therefore be viewed as composed of two uncorrelated components. One is unaffected by the point-scatterer and its relative fraction is X_{un} while the other is affected and its relative fraction is $1 - X_{un}$. The NNSD of a spectrum which is composed of several uncorrelated level sequences was computed by Berry and Robnik [6] and is applied to the circle billiard with the point-scatterer in what follows.

The unaffected spectrum consists of many levels with different angular momentum quantum numbers and thus its statistics are Poissonian [1,6]. Since the density of levels in this sequence is X_{un} , and the radius was chosen so that the total level density is unity, its NNSD is

$$P_1(S) = X_{un} e^{-X_{un}S}. \quad (12)$$

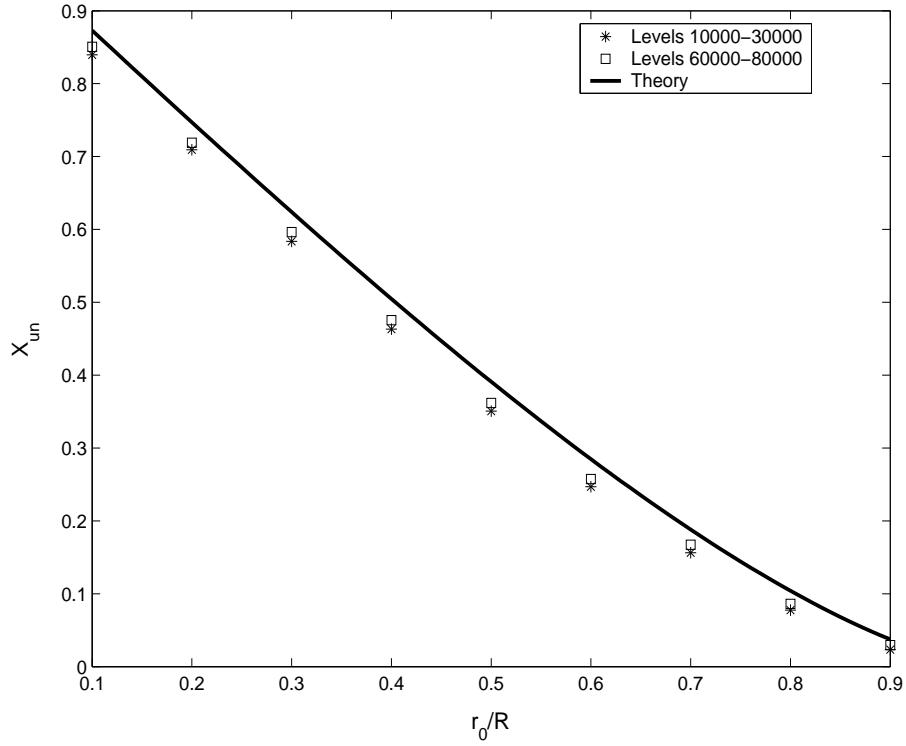


Figure 1. The fraction of unaffected energy levels for two energy windows (stars - levels 10000 to 30000; squares - levels 60000 to 80000) compared to the theoretical prediction of Eq. (11) (solid line).

The second level sequence contains the levels which are influenced by the point-scatterer and their density is $1 - X_{un}$. The exact form of its NNSD is unknown and an exact computation of this NNSD is complicated and beyond of the scope of this letter. Instead, following experience with other systems [9, 19], we will *assume* that the NNSD can be *approximated* by a semi-Poisson distribution, that is,

$$P_2(S) = 4(1 - X_{un})^2 S e^{-2(1 - X_{un})S}. \quad (13)$$

This distribution exhibits level repulsion at small spacings and exponentially small probability to find large spacings. In these works it was found numerically to describe the distribution of spacings reasonably well but there is no analytical justification for its use. Note that even if the semi-Poisson distribution is an approximation for the NNSD it may not approximate other spectral measures well. For instance, the form factor, which is the Fourier transform of the energy-energy correlation function, satisfies $K(0) = 1/2$ for the semi-Poisson distribution [7] while for a billiard with point-scatterer one expects to find $K(0) = 1$ [11]. In particular, for the rectangular billiard with a point-scatterer, the NNSD was computed analytically under some assumptions in [13] and was found *not* to be given by the semi-Poisson or the Poisson distributions. The nth neighbor spacing distributions were also calculated there and found to be those of the Poisson distribution at large spacings.

Following [6] the NNSD of the circle billiard with the point-scatterer, obtained by

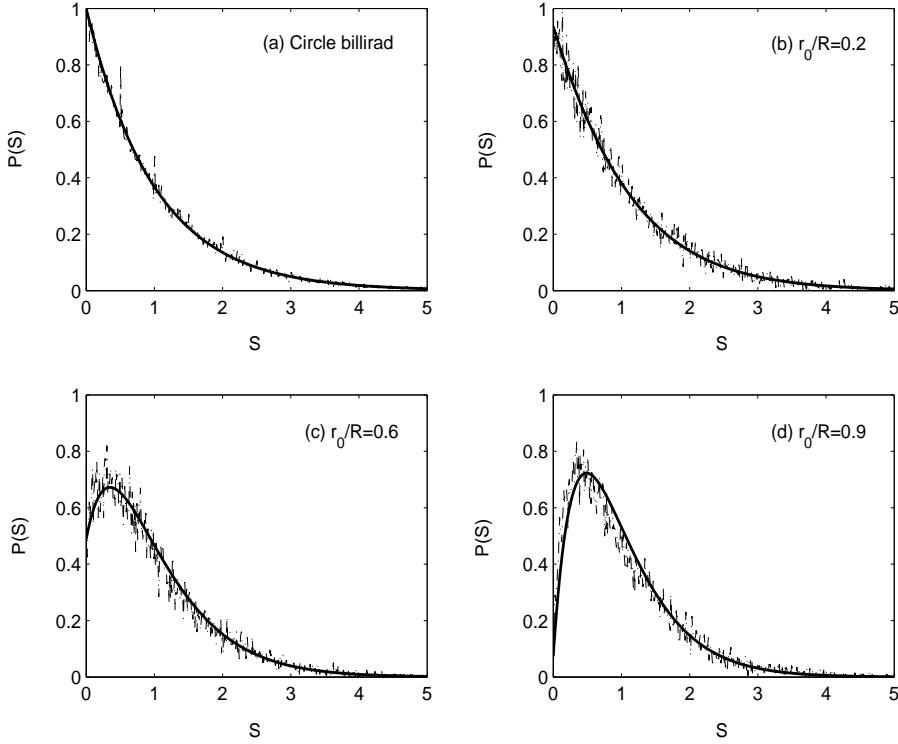


Figure 2. The theoretical nearest neighbor spacing distribution (solid line), given by equation (14), compared to numerical results (dashed-dotted line) for: (a) the circle billiard in absence of the scatterer, and with a scatterer at (b) $r_0/R = 0.2$, (c) $r_0/R = 0.6$ and (d) $r_0/R = 0.9$.

superposing the two sequences, is given by

$$P(S) = [X_{un}(2 - X_{un}) + (1 - X_{un})(2 - X_{un})^2 S] e^{-(2 - X_{un})S}. \quad (14)$$

When the perturbation is at the center, $X_{un} \rightarrow 1$, and $P(S)$ approaches the Poisson distribution. Alternatively, when the perturbation is near the boundary, and $X_{un} \rightarrow 0$, the NNSD approaches the semi-Poisson distribution. Note that the distribution (14) does not exhibit complete level repulsion since its value at $S = 0$, $P(0) = X_{un}(2 - X_{un})$, does not vanish and it is a manifestation of the existence of an infinite class of states that are unaffected by the perturbation. The NNSD of (14) is compared to numerical results in figure 2. The NNSD was computed using the levels 40000-60000 for three locations of the point-scatterer as well as for the circle billiard without the perturbation. It is clear that the agreement is very good. The main features of the distribution are captured by the simple argument leading to (14). There are slight deviations from the predictions of equation (14), mainly at large r_0/R . These can be attributed to the fact that the NNSD of the affected spectrum differs from the semi-Poisson distribution.

The results presented in figures 1 and 2 suggest that the spectrum of the circle with a point-scatterer consists of a superposition of two uncorrelated level sequences. The relative densities of these sequences are determined by the way the classical tori of the integrable system are projected into coordinate space. States for which the perturbation

is in the classically allowed region are affected while states for which the perturbation is in the classically forbidden region are nearly unaffected. We expect this behavior to be typical of systems where the scatterer affects only a fraction of the tori of the otherwise classically integrable systems. This differs from the rectangular billiard where all tori are affected by the scatterer. Another important difference compared to the rectangle billiard results of the different nature of the wavefunctions. For the rectangular billiard there are infinite classes of wavefunctions that have common zeros at rational points. Consequently, if the scatterer is placed at such a location the wavefunctions are not affected, and the distribution depends strongly on the rationality of the location of the scatterer. For the circle billiard studied in the present work, on the other hand, there is one class of eigenfunctions that vanish on the scatterer since they are antisymmetric in ϕ . These are not considered in the present work. The symmetric eigenfunctions always satisfy $\cos m\phi_0 = 1$. To obtain many functions, symmetric in ϕ , that vanish at the same location is equivalent to finding many Bessel functions which satisfy $\mathcal{J}_m(kR) = 0 = \mathcal{J}_{m+l}(k_1R)$ and $\mathcal{J}_m(\alpha kR) = 0 = \mathcal{J}_{m+l}(\alpha k_1R)$ for integer $m, l > 0$ and for $\alpha = r_0/R < 1$. Finding an infinite number of such solutions, for the same α (corresponding to the same location of the perturbation), is unlikely. However, since any Bessel function of large argument is asymptotically given by a cosine one can find states with close zeros, that is where kR and αkR are zeros of \mathcal{J}_m , while k_1R is a zero of \mathcal{J}_{m+l} and αk_1R is close to a zero of \mathcal{J}_{m+l} . In this case when the scatterer is at a zero of one of these states, the square of the wave function of the other state is much smaller there than its average value. Many such close zeros should exist to affect the spectral statistics. This question is beyond the scope of the present letter and is left for future research. Our numerical results are not sensitive enough to resolve this issue. The numerical results are used here just to verify that the mean dependence on the location of the scatterer is given by equation (14). We believe that the behavior of the circle billiard rather than that of the rectangular billiard is typical of integrable systems perturbed by a localized potential.

In summary, the spectral statistics of the circle billiard, perturbed by a point-scatterer are intermediate between those of the Poisson distribution, characteristic of integrable systems, and of the semi-Poisson distribution. The spectrum was shown to be composed of two uncorrelated components. The first contains energy levels which are nearly unaffected by the perturbation, since the point-scatterer is in a classically forbidden region where the wavefunctions are exponentially small. The relative fraction of such states was computed analytically and found to depend smoothly on the location of the point-scatterer. The second contribution is from states which are affected by the perturbation. The exact statistics of this level sequence are complicated but can be approximated by the semi-Poisson statistics. The nearest neighbor spacing distribution results of combination of the two and is a manifestation of the Berry-Robnik statistics. Other integrable systems should also exhibit qualitatively similar behavior when a localized perturbation is added to them.

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